Moment-based approximation with finite mixed Erlang distributions

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0. Outline

- Introduction
- Definitions and Notations
- Mixed Erlang distribution
- Moment-based approximation methods
- Approximation method based on finite mixed Erlang distributions: 2 contexts
- Numerical examples

1. Introduction

- We consider a positive continuous rv S for which we know the first m moments.
- Applications in actuarial science and quantitative risk management:
  - S = aggregate claim amount for a portfolio of insurance risks
  - S = aggregate claim amount for a line of business
  - S = aggregate losses for a portfolio of investment risks (e.g. credit risks)
- Main objective: evaluate cdf of S i.e. F_S
- Impossible or very difficult to find F_S analytically
- Possible to use aggregation methods:
  - Methods based on recursive numerical methods
  - Methods based on MC simulation
- May be very time-consuming to find numerically F_S
- A **moment-based approximation** can be used

2. Definitions and notations

- S : rv with cdf F_S
- j^{th} raw moments : $\mu_j(S) = E\left[S^j\right]$, $j \in \mathbb{N}^+$
- Risk measure VaR
  - $VaR_\kappa(S) = F_S^{-1}(\kappa)$, for $\kappa \in (0,1)$, where $F_S^{-1}(u) = \inf\{x \in \mathbb{R} : F_S(x) \geq u\}$
- Risk measure TVaR
  - $TVaR_\kappa(S) = \frac{1}{1-\kappa}\int_\kappa^1 VaR_u(S)\,\mathrm{d}u$ for $\kappa \in (0,1)$
  - $TVaR_\kappa(S) = \frac{E\left[S \times 1_{\{S > VaR_\kappa(S)\}}\right] + VaR_\kappa(S)(F_S(VaR_\kappa(S)) - \kappa)}{1-\kappa}$
  - If the rv S is continuous, $TVaR_\kappa(S) = \frac{E\left[S \times 1_{\{S > VaR_\kappa(S)\}}\right]}{1-\kappa}$
- Stop-loss premium :
  $\pi_S(b) = E\left[\max(S-b;0)\right] = E\left[S \times 1_{\{S>b\}}\right] - bF_S(b)$

3. Mixed Erlang distribution

- Attractive features of this class of distributions: studied by e.g. Willmot & Woo (2007), Lee & Lin (2010), and Willmot & Lin (2010)
  - Illustrate the versatility of this distribution to model claim amounts
  - Illustrate the faisability to obtain closed-form expressions for various quantities of interest in risk theory.
  - Provide several non trivial examples of distributions which belong to the class of mixed Erlang distributions
  - Provide a detailed procedure to express e.g. mixtures of exponentials, generalized Erlang distributions in terms of mixed Erlang distributions
- Closed under various operations such as convolutions, Esscher transformations (risk aggregation and ruin problems).
- Risk measures VaR, TVaR and stop-loss premium are easily obtained.

3. Mixed Erlang distribution

- Tijms (1994):
  - class of mixed Erlang distributions is dense in the set of all continuous and positive distributions
  - any nonnegative continuous distribution may be approximated by an Erlang mixture to any given accuracy
  - **Theorem.** Let F be the cdf of a positive rv. For any given $h > 0$, define the cdf F_h by

$$F_h(x) = \sum_{j=1}^{\infty} \left(F(jh) - F((j-1)h)\right) H\left(x; j, \frac{1}{h}\right), \qquad x \geq 0,$$

    where $H(x; n, \beta)$ is the Erlang cdf. Then, for any continuity point x of F,

$$\lim_{h \to 0} F_h(x) \to F(x).$$

- **Moment-based approximation** based on class of mixed Erlang dist.

3. Mixed Erlang distribution

- W : mixed Erlang rv with common rate parameter β
- Cdf of W : $F_W(y) = \sum_{k=1}^{l} \zeta_k H(x; k, \beta)$, l finite or infinite
- $H(x; k, \beta) = 1 - e^{-\beta x} \sum_{i=0}^{k-1} \frac{(\beta x)^i}{i!}$: cdf of Erlang rv of order k
- ζ_k : non-negative mass probability associated to the kth Erlang distribution in the mixture
- $\sum_{k=1}^{\infty} \zeta_k = 1$

- Representation of the mixed Erlang distribution as a compound distribution with discrete primary distribution $\{\zeta_k\}_{k=1}^{l}$ and secondary exponential distribution with rate parameter β :

$$W = \sum_{k=1}^{M} C_k,$$

  - $C_k \sim Exp(\beta)$ $(k = 1, 2, ...)$
  - M = discrete r.v. with pmf $f_M(k) = \Pr(M = k) = \zeta_k$, $k \in \mathbb{N}^+$.

- j^{th} raw moment: $\mu_j(W) = \sum_{k=1}^{\infty} \zeta_k \frac{\prod_{i=0}^{j-1}(k+i)}{\beta^j}$
- $CV(W) > \frac{1}{\sqrt{l}}$

3. Mixed Erlang distribution

- No general closed-form expression for *Value-at-risk* but easily obtained with simple numerical optimization method

$$VaR_\kappa(W) = F_W^{-1}(u) = \inf\{x \in \mathbb{R} : F_W(x) \geq u\}, \; \kappa \in (0,1)$$

- Explicit expression for *Tail Value-at-risk*:

$$\begin{aligned} TVaR_\kappa(W) &= \frac{1}{1-\kappa}\int_\kappa^1 VaR_u(W)\,\mathrm{d}u, \; \kappa \in (0,1) \\ &= \frac{1}{1-\kappa}\sum_{k=1}^{\infty} \zeta_k \frac{k}{\beta}\overline{H}(VaR_\kappa(W); k+1, \beta). \end{aligned}$$

- Explicit expression for *Stop-loss premium*:

$$\begin{aligned} \pi_W(b) &= E[\max(W-b; 0)] \\ &= \frac{1}{1-\kappa}\sum_{k=1}^{\infty} \zeta_k \left(\frac{k}{\beta}\overline{H}(b; k+1, \beta) - b\overline{H}(b; k, \beta)\right). \end{aligned}$$

4. Moment-based approximation methods

- **One approach** : approximate the unknown distribution by a mixture of known distributions.
- Several approximation methods motivated by Tijms' theorem were proposed over the years
- Examples of such methods:
  - Whitt (1982) :
    - 3 moments and 2 moments
    - $CV(S) > 1$: mixtures of 2 exponential distributions
      $F_W(x) = p_1\left(1 - e^{-\beta_1 x}\right) + p_2\left(1 - e^{-\beta_2 x}\right)$
    - $CV(S) < 1$: generalized Erlang distribution
      $F_W(x) = H(x; \beta_1, ..., \beta_r) = \sum_{i=1}^{r}\left(\prod_{l=1, l\neq i}^{r} \frac{\beta_l}{\beta_l - \beta_i}\right)\left(1 - e^{-\beta_i x}\right)$

- Continued...
  - Altiok (1985) :
    - 3 moments and 2 moments
    - $CV(S) > 1$: mixture of generalized Erlang distribution and exponential distribution $F_W(x) = pH(x; \beta_1, \beta_2) + (1-p)\left(1 - e^{-\beta_1 x}\right)$
    - $CV(S) < 1$: generalized Erlang distribution $F_W(x) = H(x; \beta_1, ..., \beta_r) = \sum_{i=1}^{r} \left(\prod_{l=1, l \neq i}^{r} \frac{\beta_l}{\beta_l - \beta_i} \right) \left(1 - e^{-\beta_i x}\right)$
  - Johnson & Taaffe (1989) :
    - 3 moments: mixture of two Erlang distributions of common order and different scale factors
    - generalize the approximation of Whitt (1982) and Altiok (1985) (for $CV(S) > 1$)
    - more than 3 moments...

4. Moment-based approximation methods

- Substantial body of literature on **3-moment** based approximations within phase-type class of distributions
- Matching first 3 moments: effective to provide a reasonable approximation (see Osogami and Harchol-Balter (2006)) but does not always suffice.
- Development of more flexible moment-based approximation methods:
  - Johnson and Taaffe (1989)
  - Horvath and Telek (2007)
  - Our proposed method.

5. Approx. method based on finite mixed Erlang distribution

- S : rv with m known moments $\mu_1(S)$, ..., $\mu_m(S)$
- Idea : map (approximate) F_S to a subclass of distributions which belongs to the class of mixed Erlang distributions
- Subclass = class of **finite** mixed Erlang distributions with
  - $F_W(y) = \sum_{k=1}^{l} \zeta_k H(x; k, \beta)$; $l < \infty$
  - $\mu_j(W) = E\left[W^j\right] = \sum_{k=1}^{l} \zeta_k \frac{\prod_{i=0}^{j-1}(k+i)}{\beta^j}$ $(j = 1, 2, ..., m)$

5. Approx. method based on finite mixed Erlang distribution

- Consider a set of first m moments $(\mu_1, ..., \mu_m) = (\mu_1(W),\mu_2(W),...,\mu_m(W))$ and $A_l = \{1, 2, ..., l\}$
- $\mathcal{ME}(\mu_1, ..., \mu_m, A_l)$: set of all **finite** mixtures of Erlang with cdf $F(y) = \sum_{k=1}^{l} \zeta_k H(x; k, \beta)$ and first m moments $(\mu_1,\mu_2,...,\mu_m)$.
- Identification of all solutions to the problem:

$$\mu_j(S) = \sum_{k=1}^{l} \zeta_k \frac{\prod_{i=0}^{j-1}(k+i)}{\beta^j}, j = 1, ..., m.$$

- Constraints: $\beta, \{\zeta_k\}_{k=1}^{l}$ are non-negative and $\sum_{k=1}^{l} \zeta_k = 1$.

5. Approx. method based on finite mixed Erlang distribution

- $\mathcal{ME}^{res}(\mu_1, ..., \mu_m, A_l)$:
  - (restricted) subset of $\mathcal{ME}(\mu_1, ..., \mu_m, A_l)$ such that **at most** m of the mixing weights $\{\zeta_k\}_{k=1}^{l}$ are non-zero.
  - Propose to use it as our class of approximations.
  - $\mathcal{ME}^{res}(\mu_1, ..., \mu_m, A_{l_1}) \subseteq \mathcal{ME}^{res}(\mu_1, ..., \mu_m, A_{l_2})$ for $l_1 \leq l_2$.
  - Members are identified by rewriting moment expressions in matrix form

5. Approx. method based on finite mixed Erlang distribution

- Obtain **all** sets of m atoms $\{i_k\}_{k=1}^m$ $(i_1 < i_2 < ... < i_m < l)$ in $A_l = \{1, 2, ..., l\}$
- For a given set of atoms $\{i_k\}_{k=1}^m$, $\mu_j(S) = \sum_{k=1}^{l} \zeta_k \frac{\prod_{i=0}^{j-1}(k+i)}{\beta^j}$ $j = 1, ..., m$ can be written as:

$$\mathbf{A}_{m,m}^T \zeta_m = \mathbf{M}\beta$$

- $\boldsymbol{\zeta}_m^T = (\zeta_{i_1}, \zeta_{i_2}, ..., \zeta_{i_m})$, $\mathbf{M} = diag(\mu_1, \mu_2, ..., \mu_m)$, $\boldsymbol{\beta}^T = (\beta, \beta^2, ..., \beta^m)$
- $\mathbf{A}_{m_1,m_2} = \begin{pmatrix} i_1 & i_2 & \cdots & i_{m_1} \\ i_1(i_1+1) & i_2(i_2+1) & \cdots & i_{m_1}(i_{m_1}+1) \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{i=0}^{m_2-1}(i_1+i) & \prod_{i=0}^{m_2-1}(i_2+i) & \cdots & \prod_{i=0}^{m_2-1}(i_{m_1}+i) \end{pmatrix}$
- m_1 : number of Erlang terms and m_2 : number of moments

5. Approx. method based on finite mixed Erlang distribution

- $\zeta_m = \left[\mathbf{A}_{m,m}^{-1}\mathbf{M}\right]\boldsymbol{\beta}$ under the constraint that $\mathbf{e}^T\zeta_m = 1$, with $\mathbf{e}$ a column vector of 1s.
- $\mathbf{e}^T \left[\mathbf{A}_{m,m}^{-1}\mathbf{M}\right]\boldsymbol{\beta}$: polynomial of degree m in β.
- Look for positive solutions in β to $\mathbf{e}^T \left[\mathbf{A}_{m,m}^{-1}\mathbf{M}\right]\boldsymbol{\beta} = \mathbf{1}$.
- Complete mixed Erlang representations via identification of mixing weights through $\zeta_m = \left[\mathbf{A}_{m,m}^{-1}\mathbf{M}\right]\boldsymbol{\beta}$.
- Repeat procedure for all possible sets of atoms.

5. Approx. method based on finite mixed Erlang distribution

- Criteria of quality among all legitimate candidates in $\mathcal{ME}^{res}(\mu_1, ..., \mu_m, A_l)$: Kolmogorov-Smirnov (KS) distance
- KS distance for two rv's S and W (with respective cdf F_S and F_W):

$$d_{KS}(S, W) = \sup_{x \geq 0} |F_S(x) - F_W(x)|.$$

- Denote by $F_{W_{m,l}}$ this approximation:

$$d_{KS}(S, W_{m,l}) = \inf_{F_W \in \mathcal{ME}^{res}(\mu_1, ..., \mu_m, A_l)} \sup_{x \geq 0} |F_S(x) - F_W(x)|,$$

where $W_{m,l}$ is a rv with cdf $F_{W_{m,l}}$.

6. Numerical examples

- **Example #1: Weibull rv S**
- $F_S(x) = 1 - \exp\left\{-(x/\beta)^{\tau}\right\}$ for $x, \tau, \beta > 0$.
- Parameters: $\tau = 1.5$ and $\beta = \Gamma(5/3)$
- $CV = 0.6790$.
- Consider class of mixed Erlang distributions $\mathcal{ME}^{res}(\mu_1, ..., \mu_m, A_{20})$ with $A_{20} = \{1, 2, ..., 20\}$

- Cardinalities of $\mathcal{ME}^{res}(\mu_1, ..., \mu_m, A_{20})$:

m	Cardinality $\mathcal{ME}^{res}(\mu_1, ..., \mu_m, A_{20})$
3	298
4	577
5	1010

- Resulting mixed Erlang approximations for $m = 3, 4$:

$$\begin{aligned} F_{W_{3,20}}(x) &= 0.0564H(x; 1, 3.0114) + 0.4097H(x; 2, 3.0114) \\ &+0.5339H(x; 4, 3.0114), \end{aligned}$$

$$\begin{aligned} F_{W_{4,20}}(x) &= 0.0355H(x; 1, 3.9083) + 0.2777H(x; 2, 3.9083) \\ &+0.4966H(x; 4, 3.9083) + 0.1901H(x; 7, 3.9083), \end{aligned}$$

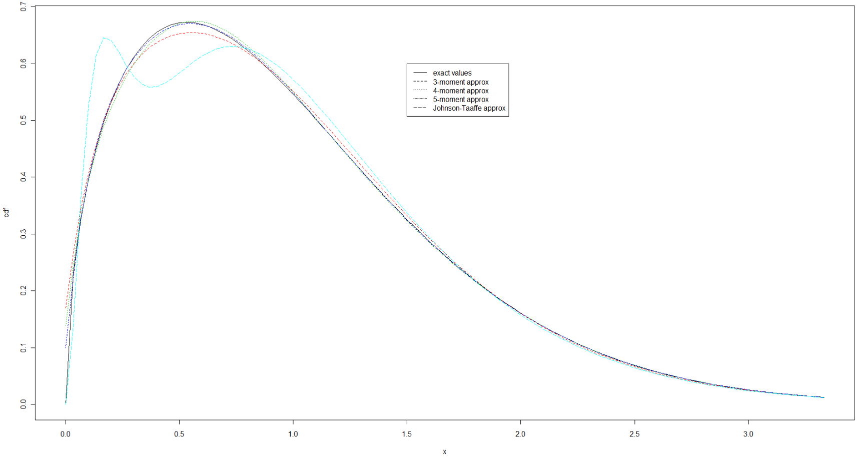
- Kolmogorov-Smirnov distances:

m	$d_{KS}\left(S, W_{m,20}\right)$
3	0.0042
4	0.0020
5	0.0007

- Quality of the approximation improves from 3 to 5 moments.

6. Numerical examples

- Comparison of pdfs of $W_{3,20}$, $W_{4,20}$, $W_{5,20}$ and S. The 3-moment approximation of Johnson and Taffee (1989) is also provided.



- Examine the tail fit: VaR and TVaR for the exact and approximated distributions

κ	$VaR_\kappa(W_{3,20})$	$VaR_\kappa(W_{4,20})$	$VaR_\kappa(W_{5,20})$	$VaR_\kappa(S)$
0.9	1.9224	1.9334	1.9324	1.9316
0.99	3.0670	3.0647	3.0651	3.0662
0.999	4.0823	4.0116	4.0146	4.0178
0.9999	5.0375	4.8706	4.8734	4.8674

κ	$TVaR_\kappa(W_{3,20})$	$TVaR_\kappa(W_{4,20})$	$TVaR_\kappa(W_{5,20})$	$TVaR_\kappa(S)$
0.9	2.4287	2.4370	2.4361	2.4354
0.99	3.5112	3.4807	3.4823	3.4844
0.999	4.4988	4.3871	4.3901	4.3897
0.9999	5.4378	5.2227	5.2244	5.2083

- **Example #2: Lognormal rv** S
- $S = \exp(\nu + \sigma Z)$ where Z is a standard normal rv
- Consider Example 5.4 of Dufresne (2007) where $\nu = 0$ and $\sigma^2 = 0.25$.
- $CV = 0.5329$.
- Lognormal has a heavier tail than mixed Erlang: no guarantee that our mixed Erlang approximation would perform well, especially for tail risk measures.

6. Numerical examples

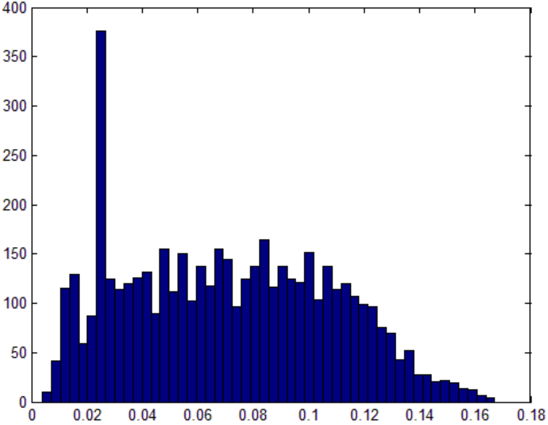
- Consider class of mixed Erlang distributions $\mathcal{ME}^{res}(\mu_1, ..., \mu_m, A_{50})$.
- Kolmogorov-Smirnov distances:

m	$d_{KS}(S, W_{m,50})$
3	0.0040
4	0.0018
5	0.0025

- KS distance increases from the 4-moment to the 5-moment approximation.
- **Remark:** $F_{W_{5,50}}$ uses Erlang-50 cdf, where 50 is the upper boundary point of A_{50} : believe that a mixed Erlang approximation with a KS distance lower than 0.0018 could be found by increasing the value l in set A_l.

6. Numerical examples

- Histograms of the KS distance for all the mixed Erlang distributions in $\mathcal{ME}^{res}(\mu_1, ..., \mu_m, A_{50})$, $m = 3, 4, 5$
- KS distances (x-axis) vs counts (y-axis)



6. Numerical examples

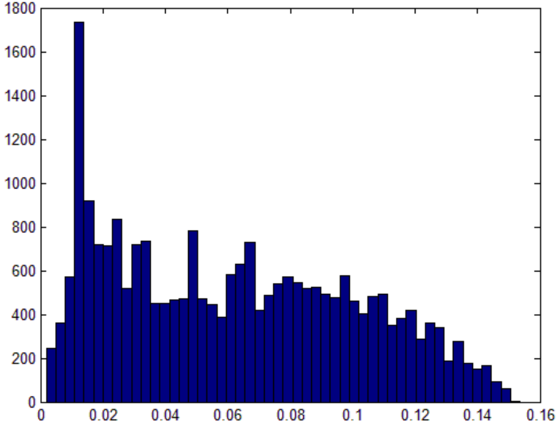


Figure: 4-moment approximations

6. Numerical examples

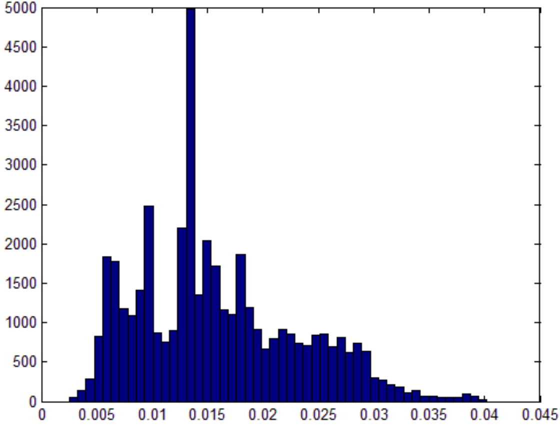
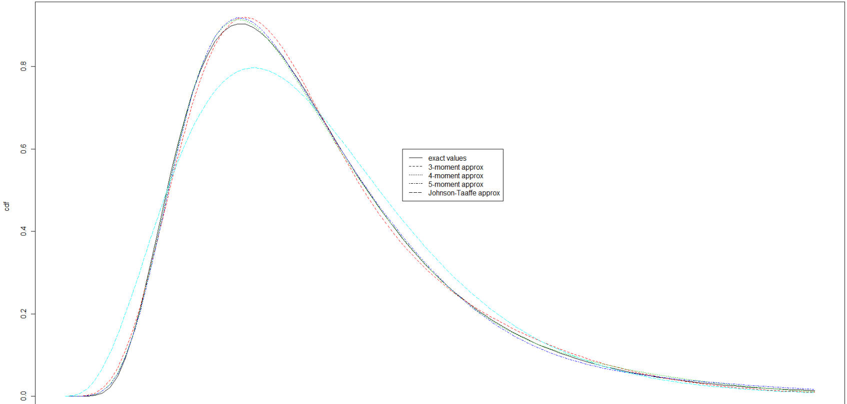


Figure: 5-moment approximations

- Overall quality of the approximations (judged by values and dispersion of KS distances) increases with number of moments matched.
- Comparison of the pdfs of $W_{3,50}$, $W_{4,50}$, $W_{5,50}$, and S. The 3-moment approximation of Johnson and Taaffe (1989) is also plotted.



6. Numerical examples

- All three mixed Erlang approximations provide an overall good fit to the exact distribution.
- To further examine the tail fit, specific values of VaR and TVaR for the exact and approximated distributions are provided below:

κ	$VaR_\kappa(W_{3,50})$	$VaR_\kappa(W_{4,50})$	$VaR_\kappa(W_{5,50})$	$VaR_\kappa(S)$
0.9	1.9129	1.9056	1.8891	1.8980
0.99	3.1223	3.0991	3.2220	3.2001
0.999	4.9237	5.0623	4.3746	4.6885
0.9999	6.0352	6.2642	6.8895	6.4206

κ	$TVaR_\kappa(W_{3,50})$	$TVaR_\kappa(W_{4,50})$	$TVaR_\kappa(W_{5,50})$	$TVaR_\kappa(S)$
0.9	2.4540	2.4550	2.4697	2.4616
0.99	3.9008	3.8625	3.7700	3.8413
0.999	5.4245	5.5431	5.5204	5.4341
0.9999	6.4189	6.6093	7.4023	7.2879

6. Numerical examples

- *VaR* and *TVaR* values of our mixed Erlang approximations compare reasonably well to their lognormal counterparts.
- Improvement is not monotone with the number of moments matched: well known that increasing the number of moments does not necessarily lead to a higher quality approximation in moment-matching techniques.

- **Example #3: Real data**
- Normalized damage amounts from 30 most damaging hurricanes in United States from 1925 to 1995 (provided by Pielke and Landsea (1998) and analyzed by Brazauskas et al. (2009)).
- Purpose of this example: not to carry an exhaustive statistical analysis of this dataset, but provide a simple fit with a finite mixed Erlang distribution
- First 4 empirical moments:

j	1	2	3	4
μ_j	11.7499	317.5154	15604.47	986686.4

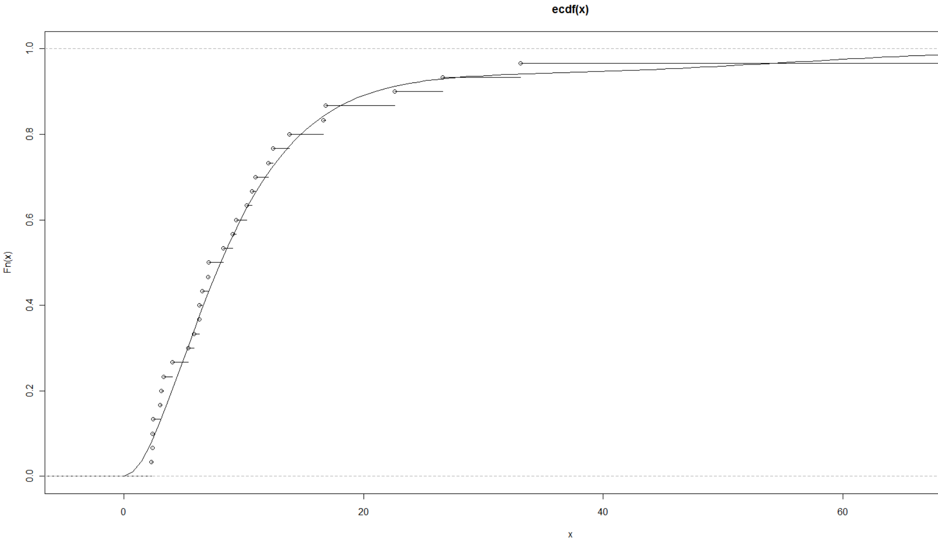
- $CV = 1.1401$.

- Perform the approximation with $\mathcal{ME}^{res}(\mu_1, ..., \mu_m, A_{30})$ for $m = 3$ and 4.
- Kolmogorov-Smirnov distances (with empirical distribution) :

m	$d_{KS}(S, W_{m,30})$
3	0.0773
4	0.0769

- Critical value of the KS hypothesis test at a significance level of 1%: $1.63/\sqrt{30} = 0.2976$
- Do not reject both distributions as a plausible model for the dataset.

6. Numerical examples



7. Moment-based approx. with known rate parameter

- Slightly different context.
- Distribution function F_S is known to be of mixed Erlang form with known $\beta > 0$ and $(\mu_1, \mu_2, ..., \mu_m)$.
- Distribution itself unknown or difficult to evaluate.
- Restrict to sets of **finite** mixture of Erlang distributions.
- Bounds on risk measures can be established.
- Connection with extremal points of a discrete moment-matching problem.

7. Moment-based approx. with known rate parameter

- $F_S \in \mathcal{ME}(\mu_1, ..., \mu_m, \beta)$: set of all mixed Erlang dist. for $l = \infty$, rate parameter β and first m moments $(\mu_1, ..., \mu_m)$.
- $\mathcal{ME}(\mu_1, ..., \mu_m, A_l, \beta)$: subset of $\mathcal{ME}(\mu_1, ..., \mu_m, \beta)$ for a given $l \in \mathbb{N}^+$.
- $\mathcal{ME}^{ext}(\mu_1, ..., \mu_m, A_l, \beta)$: subset of $\mathcal{ME}(\mu_1, ..., \mu_m, A_l, \beta)$ such that **at most** $(m+1)$ of mixing weights $\{\zeta_k\}_{k=1}^{l}$ are non-zero.
- Consider two approaches to derive bounds on $E[\phi(S)]$ for ϕ a given function (such that expectation exists):
  - Based on discrete s-convex extremal distributions
  - Based on moment bounds on discrete expected stop-loss transforms

8. Discrete s-convex extremal distributions

- $D(\alpha_1, ..., \alpha_m, A_l)$: all discrete dist. with support A_l with first m moments $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_m)$.
- $D^{ext}(\alpha_1, ..., \alpha_m, A_l)$: all discrete dist. with support A_l with at most $(m+1)$ non-zero mass points with first m moments are $\boldsymbol{\alpha}$.
- For a given $\beta > 0$: one-to-one correspondence between discrete classes and mixed-Erlang classes
- Each dist. in $D(\alpha_1, ..., \alpha_m, A_l)$ (and $D^{ext}(\alpha_1, ..., \alpha_m, A_l)$) corresponds to a mixed Erlang dist. in $\mathcal{ME}(\mu_1, ..., \mu_m, A_l, \beta)$ (and $\mathcal{ME}^{ext}(\mu_1, ..., \mu_m, A_l, \beta)$) (see De Vylder 1996)
- Allows to use theory on sets of discrete distributions e.g. in Prékopa (1990), Denuit, Lefèvre and Mesfioui (1999), Courtois et. al (2006).

- **Definition s-convex:** Let C be a subinterval of $\mathbb{R}$ or a subset of $\mathbb{N}$ and ϕ a function on C. For two rv's X and Y defined on C, X is said to be smaller than Y in the s-convex sense, namely $X \preceq_{s-cx}^{C} Y$, if $E[\phi(X)] \leq E[\phi(Y)]$ for all s-convex functions ϕ.
- Examples of s-convex functions: $\phi(x) = x^{s+j}$ and $\phi(x) = \exp(cx)$ for $c \geq 0$.
- $K_{s,\min}$ and $K_{s,\max}$: s-extremum rv's on $D(\alpha_1, ..., \alpha_m, A_l)$

$$E[\phi(K_{s,\min})] \leq E[\phi(K)] \leq E[\phi(K_{s,\max})]$$

for any s-convex function ϕ and any $K \in D(\alpha_1, ..., \alpha_m, A_l)$.

- General distribution forms of $K_{s,\min}$ and $K_{s,\max}$ are given in Prékopa (1990) and Courtois et al. (2006)
- $W_K = \sum_{j=1}^{K} C_j$ be a mixed Erlang rv.
- Denuit, Lefèvre and Utev (1999) state that the s-convex order is stable under compounding.
- **Lemma:** If $K \preceq_{s-cx}^{A_l} K'$', then $W_K \preceq_{s-cx}^{\mathbb{R}^+} W_{K'}$.
- Can apply this Lemma to $W_{K_{s-\min}}$ and $W_{K_{s-\max}}$:

$$W_{K_{s-\min}} \preceq_{s-cx}^{\mathbb{R}^+} W_K \preceq_{s-cx}^{\mathbb{R}^+} W_{K_{s-\max}}$$

- Allows to find general distribution forms of $F_{W_{K_{s-\min}}}$ and $F_{W_{K_{s-\max}}}$
- For s-convex functions $\phi(x) = x^{s+j}$ and $\phi(x) = \exp(cx)$,can obtain bounds:

$$\begin{aligned} E\left[W_{K_{s-\min}}^{s+j}\right] &\leq E\left[W_K\right] \leq E\left[W_{K_{s-\max}}^{s+j}\right] \\ E\left[\exp(cW_{K_{s-\min}})\right] &\leq E\left[\exp(cW_K)\right] \leq E\left[\exp(cW_{K_{s-\max}})\right] \end{aligned}$$

9. Moment bounds on discrete expected stop-loss transforms

- Extrema with respect to s-convex order allows to derive bounds on $E\left[\phi(S)\right]$ for all s-convex functions ϕ.
- Approach not appropriate to derive bounds for TVaR and stop-loss premium when $m \geq 2$.
- Use an approach (based on increasing convex order) inspired from Courtois and Denuit (2009) and Hürlimann (2002).
- **Main idea:**
  - consider $D(\alpha_1, ..., \alpha_m, A_l)$ for $m \in \{2, 3, ...\}$
  - find lower and upper bounds for $E\left[(K-k)_+\right]$ on $D(\alpha_1, ..., \alpha_m, A_l)$ for all $k \in A_l$
  - from lower (upper) bound, derive corresponding rv K_{m-low} (K_{m-up})
  - $E\left[(K_{m-low}-k)_+\right] \leq E\left[(K-k)_+\right] \leq E\left[(K_{m-up}-k)_+\right]$ on $D(\alpha_1, ..., \alpha_m, A_l)$ for all $k \in A_l$
  - implies under the increasing convex order: $K_{m-low} \preceq_{icx} K \preceq_{icx} K_{m-up}$

9. Moment bounds on discrete expected stop-loss transforms

- Increasing convex order is stable under compounding:

$$W_{K_{m-low}} \preceq_{icx} W_K \preceq_{icx} W_{K_{m-up}}$$

- From Denuit et al. (2005):

$$TVaR(W_{K_{m-low}}) \leq TVaR(W_K) \leq TVaR(W_{K_{m-up}})$$

10. Example - Portfolio of dependent risks

- Portfolio of n dependent risks (common mixture model of Cossette and al. (2002))
- $S = X_1 + ... + X_n$: aggregate claim amount with $X_i = B_i I_i$.
- Conditional on a common mixture rv Θ with pmf p_Θ, $\{I_i\}_{i=1}^n$ are assumed to form a sequence of independent Bernoulli rv's with

$$\Pr\left(I_i = 1 \left| \Theta = \theta \right.\right) = 1 - r_i{}^\theta \text{ for } r_i \in (0, 1).$$

- B_i $(i = 1, ..., n)$ are assumed to form a sequence of iid rv's, independent of $\{I_i\}_{i=1}^{20}$ and Θ.
- B_i $(i = 1, ..., n)$: exponentially distributed with mean 1
- Distribution of S : two-point mixture of a degenerate rv at 0 and a mixed Erlang with $l = n$ and $\beta = 1$.

- Parameters:
  - $n = 20$ risks
  - Θ has a logarithmic distribution with pmf $p_{\Theta}(j) = (0.5)^{j} / (j \ln 2)$ for $j = 1, 2, ...$
  - constants r_i are set such that the (unconditional) mean of I_i is $q_i = 1 - E\left[(r_i)^{\Theta}\right]$ with $q_1 = ... = q_{10} = 0.1$ and $q_{11} = ... = q_{20} = 0.02$. It
- Perform moment-based approximation on rv $Y = (S \,|S > 0)$ rather than S
- j-th moment of Y : $\mu'_j \equiv E\left[Y^j\right] = \frac{E\left[S^j\right]}{1 - F_S(0)}$
- $CV(Y) = 0.9603$.
- Methods of Whitt (1982) and Altiok (1985) not applicable here: constraints on CV and third moment ($\mu_3 \mu_1 \geq 1.5 {\mu_2}^2$) not satisfied.
- Method of Johnson and Taaffe (1989): $r = 2$, $\beta_1 = 0.7627$, $\beta_2 = 2.8939$ and $p = 0.5742$.

- **First approach: discrete s-convex extremal distributions**
- Find cdfs $F_{W_{K_{s-\min}}}$ and $F_{W_{K_{s-\max}}}$ for $m = 4, 5$ $(s = m + 1)$
- Consider two distributional characteristics of S :
  - higher-order moments $E\left[S^j\right]$ for $j = 4, 5, 6$
  - exponential premium principle $\varphi_\eta\left(S\right) = \frac{1}{\eta}\ln E\left[e^{\eta S}\right]$ for $\eta > 0$.
- Distributions $F_{W_{K_{m+1-\min}}}$ and $F_{W_{K_{m+1-\max}}}$ provide bounds to these risk measures associated to the rv S

10. Example - Portfolio of dependent risks

- Bounds on $E\left[S^j\right]$ and $\varphi_\eta(S) = \frac{1}{\eta}\ln E\left[e^{\eta S}\right]$:

j	$E\left[W^j_{K_{5-\min}}\right]$	$E\left[W^j_{K_{6-\min}}\right]$	$E\left[S^j\right]$	$E\left[W^j_{K_{6-\max}}\right]$	$E\left[W^j_{K_{5-\max}}\right]$
4	138.7579	138.7579	138.7579	138.7579	138.7579
5	1125.9592	1129.1880	1129.1880	1129.1880	1149.9348
6	10748.5738	10873.8020	10881.2732	10922.7337	11993.6176

θ	$\varphi_\eta\left(W_{K_{5-\min}}\right)$	$\varphi_\eta\left(W_{K_{6-\min}}\right)$	$\varphi_\eta(S)$	$\varphi_\eta\left(W_{K_{6-\max}}\right)$	$\varphi_\eta\left(W_{K_{5-\max}}\right)$
0.2	1.5545	1.5546	1.5546	1.5548	1.5564
0.1	1.3536	1.3536	1.3536	1.3536	1.3536
0.01	1.2137	1.2137	1.2137	1.2137	1.2137

- Bounds get sharper as the number of moments involved increases.

10. Example - Portfolio of dependent risks

- **Second approach: moment bounds with discrete expected stop-loss transforms**
- Values of TVaR for $W_{K_{m-low}}$ and $W_{K_{m-up}}$ $(m = 4, 5)$:

	Exact	J&T	$TVaR_\kappa(...)$ for $m = 3$	
κ	$TVaR_\kappa(S)$	$TVaR_\kappa(W)$	W_{3-low}	W_{3-up}
0.9	5.0696	5.1389	4.798911	5.333275
0.95	6.2214	6.2563	5.771565	6.615174
0.99	8.8460	8.7491	7.911982	9.675684
0.995	9.9589	9.7892	8.799191	11.116631
0.999	12.5066	12.156	10.805712	15.181871

10. Example - Portfolio of dependent risks

	Exact	$TVaR_\kappa(...)$ for $m=4$		$TVaR_\kappa(...)$ for $m=5$	
κ	$TVaR_\kappa(S)$	$W_{K_{4-low}}$	$W_{K_{4-up}}$	$W_{K_{5-low}}$	$W_{K_{5-up}}$
0.9	5.0696	4.9222	5.2062	4.9800	5.1490
0.95	6.2214	5.9708	6.4548	6.0594	6.3642
0.99	8.8460	8.2899	9.3301	8.4655	9.1767
0.995	9.9589	9.2500	10.5629	9.4679	10.3775
0.999	12.5066	11.4122	13.4854	11.7323	13.1382

- Inequality verified:

$$TVaR(W_{K_{m-low}}) \leq TVaR(W_K) \leq TVaR(W_{K_{m-up}})$$

- Interval estimate of $TVaR_\kappa(S)$ shrinks as number of moments matched increases.